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APPROXIMATE SOLUTIONS OF TYPES (3, 1) AND (4)

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In linearized gravitational theory, space-time is Minkowskian, the gravitational field is the linear approximation to the curvature tensor, and its equations are the Bianchi identities of the linear approximation. This note presents a class of solutions to the linearized equations which includes fields which vanish asymptotically and are nowhere singular.

We remark that the most general self-dual solutions of types (3,1) and (4) may be expressed locally as

$$C = N \otimes F + F \otimes N - (A : F)\, N \otimes N \ , \tag{1}$$

where N, F and A are self-dual 2-forms satisfying

$$N : N = 0 = N : F, \quad N : A = 1 \ , \tag{2}$$

$$N : dN = 0, \quad dF = 0, \quad \nabla A = 0 \ , \tag{3}$$

and, for any tensors $X^{\cdots}_{\cdots}$ and $Y^{\cdots}_{\cdots}$,

$$X : Y = \frac{1}{2} X^{\cdots}_{..ij} Y^{ji..}_{....} \ . \tag{4}$$

Consider the case of constant N. We choose coordinates so that

$$g = 2dwdv - 2dzd\overline{z}, \quad N = dw \wedge dz, \tag{5}$$

and find that F takes the form

$$F = \Phi M + \Psi N, \quad M = dw \wedge dv - dz \wedge d\overline{z} \ , \tag{6}$$

where $\Phi = \Phi(w, z)$ and

$$\Psi = X(\overline{w}, z) + \overline{z}\Phi_w + v\Phi_z \ . \tag{7}$$

Taking $A = dv \wedge d\overline{z}$, we obtain the general plane fronted wave

$$C = \Phi \left(N \otimes M + M \otimes N \right) + \Psi N \otimes N \ . \tag{8}$$

We extend this solution to the complex domain by taking v, w, z, $\overline{z}$ to be independent complex variables. We then substitute

$$w = u - ia\left(\zeta\overline{\zeta} + 1\right), \quad v = -\left(r - ia\right)^{-1} \ , \tag{9}$$

$$z = \zeta, \quad \overline{z} = \left(r + ia\right)\left(r - ia\right)^{-1}\overline{\zeta} \ , \tag{10}$$

make the conformal transformation

$$g' = (r - ia)^2 g \ , \quad C' = (r - ia)C \ , \tag{11}$$

and put $M' = (r - ia)^2 M$, $N' = (r - ia)N$. Then, writing

$$\lambda = du + ia\left(\overline{\zeta}d\zeta - \zeta d\overline{\zeta}\right), \quad \nu = dr, \tag{12}$$

$$\mu = (r - ia)d\zeta, \quad \overline{\mu} = (r + ia)d\overline{\zeta} \ , \tag{13}$$

we obtain

$$g' = 2\lambda\nu - 2\mu\overline{\mu}, \quad M' = \lambda \wedge \nu - \mu \wedge \overline{\mu}, \quad N' = \lambda \wedge \mu \ , \tag{14}$$

and

$$C' = (r - ia)^{-2}\Phi\left(N' \otimes M' + M' \otimes N'\right) + (r - ia)^{-1}\Psi N' \otimes N' \ , \tag{15}$$

$$\Psi = X(w, \zeta) + \left(\frac{r + ia}{r - ia}\right)\overline{\zeta}\Phi_w - (r - ia)^{-1}\Phi_\zeta \ . \tag{16}$$

Finally, we revert to real space-time by taking the coordinates u and r to be real, ζ and $\overline{\zeta}$ to be complex conjugates. For $a = 0$ we have a class of self-dual R-T fields. For $a \neq 0$, w is nowhere zero, and the solution given by

$$\Phi = c_1 w^{-3}, \quad X = c_2 w^{-3}, \tag{17}$$

for example, is non-singular and asymptotically vanishing.